Yang-Mills Instantons in closed Robertson-Walker Space-Time†

Hilmar Forkel
Physics Department
State University of New York at Stony Brook
Stony Brook, New York, 11794.

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ABSTRACT

We construct a four-parameter class of self-dual instanton solutions of the classical SU(2)-Yang-Mills equations in a closed Euclidean Robertson-Walker space-time.

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The discovery of the first Yang-Mills instantons [1] and their fundamental physical relevance [2] initiated an intense research activity devoted to their study. It led in particular to the construction of multi-instanton solutions in continuous families with an increasing number of degrees of freedom. Finally, with the advent of the ADHM method [3] an explicit construction principle for the most general self-dual solutions has been given, and as a consequence [4] all stable finite-action solutions of the Euclidean Yang-Mills equations in flat (compactified) space-time are known today.

For instantons in curved space-time, however, the situation is quite different. No general classification exists and not many explicit solutions are known. Soon after the first flat space-time solution a straightforward generalization to conformally flat de-Sitter space-times was found using the conformal invariance of the self-duality equation [5],[6]. Shortly afterwards a one-instanton solution in the Schwartzschild geometry, suitably continued to Euclidean signature, was constructed [7]. It showed in particular that even very weak gravitational fields can have a nontrivial influence on the general topological structure of instantons. Further explicit Yang-Mills instanton solutions are known in S^4 space-times [8] and in the Taub-NUT metric [9].

In this paper we are looking for new instantons in a curved Euclidean space-time with topology $S^3 \otimes R$. We first discuss the general implications of this topology for structure and classification of the instantons. We then construct a four-parameter class of explicit solutions of the self-duality equation for the Yang-Mills field strength and summarize with a brief discussion.

The topological structure of nontrivial classical configurations of an SU(2) gauge field A is determined by the physical requirement of a finite Euclidean Yang-Mills action

$$S = -\frac{1}{\mathrm{g}^2} \int_M tr[F \wedge *F], \qquad (1)$$

where M is a four dimensional Euclidean space-time and

$$F = dA + A \wedge A = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])\, dx^\mu \wedge dx^\nu \qquad (2)$$

is the field strength of the (SU(2)-Lie algebra valued) gauge field, $*F$ its dual and g is the Yang-Mills coupling constant. (The trace is over the internal indices.)

We choose the topological structure of space-time to be $S^3 \otimes R$ (the three compact dimensions are spacial), but do not yet fix the detailed form of the metric aside from its Euclidean signature and the topology. The spacial hypersphere at time t will be denoted S^3_t.

The finite-action condition forces the field strength F to vanish sufficiently fast in the asymptotic regions S^3_t with $t \to \pm\infty$ of space-time, in which A has consequently to become pure gauge:

$$A \to \quad U^{-1}_{\pm\infty} dU_{\pm\infty} \qquad\qquad \text{for} \quad t \to \pm\infty. \tag{3}$$

Every continuous gauge-field configuration with finite action defines in this way two mappings $U_{-\infty}, U_{+\infty}$ from the asymptotic regions $S^3_{t=-\infty}$ and $S^3_{t=+\infty}$, respectively, to SU(2) $\cong S^3$.

Both these mappings fall into homotopy classes, which form the third homotopy group of S^3, $\pi_3(S^3) = Z$, and are therefore characterized by integer degrees (winding numbers) $q_{-\infty}$ and $q_{+\infty}$. So each gauge-field configuration specifies values for both these integers, which are unchanged by any continuous deformation of the field.

Topologically nontrivial gauge transformations (which can be nonsingular for the considered space-time topology), however, may change $(q_{-\infty},\ q_{+\infty})$. Every such transformation maps S^3_t to $SU(2)$ for all t and by continuity always with the same homotopy degree . A gauge transformation with nonzero degree changes the asymptotic form of A, eq. (3), and in particular adds its degree to the degrees of $U_{\pm\infty}$. This leads to the conclusion that only the difference of these degrees, $Q = q_{+\infty}$ - $q_{-\infty}$, characterizes a field configuration in a gauge-independent way. Indeed, $-Q$ can be identified with the integral of the second Chern class of F over M,

$$Q = -\frac{1}{8\pi^2}\int_M tr[F \wedge F], \tag{4}$$

which is exactly the quantity used for the topological classification of gauge fields in flat space-time. It is gauge invariant and well defined for the asymtotically decaying A's under consideration. In order to show its equivalence with the first definition of Q explicitly, we use[1]

$$tr[F \wedge F] = dtr(F \wedge A - \frac{1}{3} A \wedge A \wedge A) \tag{5}$$

together with Stoke's theorem (assuming A to be nonsingular on M) in order to rewrite Q as a surface integral over the boundary of $S^3 \otimes [-\tau, \tau]$ for $\tau \to \infty$, *i.e.* in a very large compact time interval of M,

$$Q = -\frac{1}{8\pi^2}\int_{\partial M} tr(F \wedge A - \frac{1}{3} A \wedge A \wedge A) \tag{6}$$

[1]The integrand in (4) being an exact form (total divergence) makes Q a topological invariant on a compact manifold without boundary, the second Chern number.

with

$$\partial M = S^3_{+\tau} + S^3_{-\tau} \qquad \text{for}\, \tau \to \infty. \tag{7}$$

Using the asymptotic form of A on ∂M, (3), we arrive as anticipated at

$$Q = q_{+\infty} - q_{-\infty}, \tag{8}$$

where

$$q_{\pm\infty} = \lim_{\tau \to \pm\infty} \frac{1}{24\pi^2} \int_{S^3_\tau} tr[U^{-1}_{\pm\infty} dU_{\pm\infty} \wedge U^{-1}_{\pm\infty} dU_{\pm\infty} \wedge U^{-1}_{\pm\infty} dU_{\pm\infty}] \tag{9}$$

is the well-known expression for the homotopic degree of the mappings $U_{\pm\infty}$, respectively. The gauge invariance of Q can now be checked explicitly by using the behaviour of (9) under gauge transformations.

We are now going to construct explicit instanton solutions of the Yang-Mills equations in the $Q = 1$ sector, which provide an example for the general asymptotic structure of the gauge field and its topological classification. In particular we are interested in solutions with self-dual field strength

$$F = {*F}, \tag{10}$$

which correspond to absolute minima of the action[2]. Note, however, that not all stable finite-action solutions of the Yang-Mills equations have to be necessarily self-dual in curved space-time.

To find solutions of (10) with $Q = 1$, we start with the ansatz

$$A = f(t)\, U_1^{-1} dU_1, \qquad U_1 = \cos\psi + i\tau_a \hat{r}^a(\theta, \phi) \sin\psi \tag{11}$$

(θ, ϕ and ψ are polar coordinates on the spacial S^3 of M, $\hat{r}^a$ are the cartesian components of the unit vector in R^3 and τ_a are the Pauli matrices), which preserves the rotational (equivalent to constant gauge-) symmetry of the asymptotic form (3). Note that this ansatz gives A in temporal (Weyl-) gauge, $A_0 = 0$. U_1 is (topologically) the identity map from the spacial S^3 to SU(2). Inserting (11) into (10), the self-duality condition becomes

$$\dot{f} L_i = \frac{1}{2} f(f-1) \epsilon_{ijk} [L^j, L^k]. \tag{12}$$

[2]By using Schwarz's inequality for the inner product of Lie-algebra valued 2-forms on M one establishes as in flat space the lower bound $S \geq 8\pi^2 |Q|/\mathrm{g}^2$ for the action in the topological sector Q, which is realized by self-dual fields.

The latin indices ($i = 1, 2, 3$) refer to the spacial coordinates on M, the dot indicates the time derivative and $L_i = U_1^{-1}\partial_i U_1 = \theta^a{}_i\, i\tau_a$ are the components of the left-invariant Cartan-Maurer form, pulled back by the identity map U_1. (ϵ_{ijk} are the components of the Levi-Civita (volume) form on the spacial hypersphere.)

If (12) is to hold for all times, the time independence of the L_i's forces

$$\alpha = f(f-1)/\dot{f} \tag{13}$$

to be constant. The nonsingular solution of this differential equation,

$$f(t) = \frac{1}{1 + \exp{(t-t_0)/\alpha}}, \tag{14}$$

determines the time dependence of the instanton. The integration constant t_0 reflects the translational invariance of the action (1) in time direction.

After decomposing the spacial part of the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 + d\sigma^2 \tag{15}$$

on M into an orthonormal dreibein field,

$$d\sigma^2 = \gamma_{ij}dx^i dx^j, \qquad \gamma_{ij} = \delta_{ab}e^a{}_i e^b{}_j, \tag{16}$$

(latin indices from the beginning of the alphabet label the axis of the orthonormal frame), eqn. (12) can be rewritten as

$$[L_a, L_b] = \alpha^{-1}\epsilon_{ab}{}^c L_c. \tag{17}$$

The matrices $\alpha L_a(x) = O_a{}^b(x)\frac{\tau_b}{2i}$ must therefore form locally a representation of the basis of the SU(2) Lie algebra, which implies the orthogonality of $O_a{}^b(x) = -2\alpha e_a{}^i\theta^b{}_i$. Redefining the dreibein by a rotation with O^{-1} (which doesn't affect the decomposition of the metric (16)), its inverse becomes proportional to the pulled-back Cartan-Maurer form:

$$e^a{}_i = -2\alpha\theta^a{}_i. \tag{18}$$

Now the θ's are the pull-back of an orthonormal frame for the natural left-invariant metric on SU(2) (the three-dimensional geometrical unit sphere), which are still proportional to an orthonormal frame on the spatial hypersphere, as (18) shows. This implies that the spacial metric on M is that of a *geometrical* three-sphere, too. Indeed,

$$d\sigma^2 = 4\alpha^2\delta_{ab}\theta^a{}_i\theta^b{}_j\, dx^i dx^j = 4\alpha^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{19}$$

is the metric of a three-dimensional sphere with radius $\mathcal{R} = 2|\alpha|$ in polar coordinates. The complete space-time metric is therefore the Euclidean analog of a (homogenous and isotropic) Robertson-Walker line element with a constant spacial radius.

The above derivation shows that solutions of the self-duality equation of the form (11) in space-times with topological structure $S^3 \otimes R$ exist only if space is a static geometrical hypersphere. The metric (15) is therefore uniquely determined to be the Euclidean version of the Einstein universe [10].

The preceding discussion determined only the modulus of α, $|\alpha| = \mathcal{R}/2$. Inspection of (13) and (18) shows that the two possible signs interchange the asymptotic values of f as well as the orientation of the ${e^a}_i$ and lead to two different instanton solutions[3]. They can, however, be transformed into each other by U_1, eq. (11), taken as a (topologically nontrivial) gauge transformation. We will choose the solutions with negative α as the representatives of their gauge-equivalence class and denote them as A^+.

Eq. (12) shows that anti-self-dual solutions A^-, the anti-instantons with $Q = -1$, can be obtained from the instanton solutions by changing α to $-\alpha$ in the definition of f, eq. (14).

Implementing these remarks, the explicit form of the solutions in a closed, static Robertson-Walker space-time of radius $\mathcal{R}$ finally becomes

$$A^{\pm} = \frac{1}{1 + \exp \mp 2(t - t_0)/\mathcal{R})} U_1^{-1} dU_1 \tag{20}$$

with U_1 as in (11).

They form a continuous four-dimensional family, parametrized by their center in time direction, t_0, and three parameters corresponding to a global SU(2) right transformation, which merely changes the (arbitrary) relative position of the poles on S^3_{space} and $S^3_{SU(2)}$.

The field strength of these (anti-) instantons, expressed in the orthonormal frame, is

$$F^{\pm} = \dot{f} \frac{\tau_a}{2i\alpha} (e^0 \wedge e^a \pm \frac{1}{2} {\epsilon^a}_{bc} \, e^b \wedge e^c), \tag{21}$$

where $e^a = {e^a}_i dx^i$ and $e^0 = dt$. Note that the extension of the instanton in time $|\alpha| \sim \mathcal{R}$ as well as in space is gouverned by $\mathcal{R}$. In contrast to the flat space case, the metric introduces a dimensionful parameter which fixes the scale of the solutions.

[3]This is a consequence of the PT invariance of the Yang-Mills action and its spontaneous breaking by the instanton.

The second Chern class of the (anti-) instantons,

$$c_2^{\pm} = \frac{1}{8\pi^2} tr[F^{\pm} \wedge F^{\pm}] = \mp \frac{\dot{f}^2}{16\pi^2\alpha^2} \epsilon_{abc} e^0 \wedge e^a \wedge e^b \wedge e^c, \tag{22}$$

can be used to check their instanton number explicitly:

$$Q^{\pm} = -\int_M c_2^{\pm} = \pm 3\mathcal{R} \int_{-\infty}^{\infty} \dot{f}^2 \, dt = \pm 1. \tag{23}$$

To summarize, we have presented a four-parameter family of self-dual instanton solutions of the SU(2)-Yang-Mills equation in a closed, static Robertson-Walker space-time. Because of the well-known fact that the Euclidean energy-momentum tensor of self-dual fields vanishes, they do not interact (at the classical level) with gravity. The metric (15) together with the instantons may consequently be interpreted as a self-consistent classical solution of the Einstein-Yang-Mills equations with space-time in form of a matter-dominated Euclidean Einstein universe.

Furthermore, the obtained solutions provide a new example for the influence of an even very weak gravitational space-time curvature on the qualitative structure and scale of Yang-Mills instantons. In the light of the recent investigations of Skyrmions on three-dimensional hyperspheres [11], they might also, as in flat space, be useful in providing new Skyrmion configurations and their generalization to finite temperature [12] via the Atiyah-Manton construction [13].

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